MATH 207(c)–Analysis IV

1. Find the coefficients of the Taylor series of the following functions around the given point:

(a) $f(z) = e^z$ at $z_0 = 0$.

(b) $f(z) = e^z$ at $z_0 = \pi i$.

(c) $f(z) = e^{z^2}$ at $z_0 = 0$.

(d) $f(z) = \cos(z-3)$ at $z_0 = 3$.

(e) $f(z) = \big(\sin(z)\big)^2$ at $z_0 = 0$.

2. For the functions below, compute the Laurent series at the specified point, determine the radius of convergence and specify the nature of the singularity (i.e. regular point, pole or essential singularity):

(a) $g(z) = \frac{e^z}{(z-2)^2}$ at $z_0 = 2$.

(b) $g(z) = \frac{1}{z(z+1)}$ at $z_0 = -1$.

(c) $g(z) = \frac{z^3-z+2}{(z-i)^2}$ at $z_0 = i$.

(d) $g(z) = \frac{\cos\left((z-1)^2\right)}{(z-1)^3}$ at $z_0 = 1$.

3. For the functions below, compute the Laurent series at the specified point, determine the radius of convergence and specify the nature of the singularity (i.e. regular point, pole or essential singularity):

(a) $f(z) = z\cos(\frac{1}{z})$ at $z_0 = 0$.

(b) $f(z) = \frac{\sin(z)}{(z-\pi)}$ at $z_0 = \pi$.

(c) $f(z) = \frac{z^{\frac{1}{2}}}{(z-1)^2}$ at $z_0 = 1$.

4. Let $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$ be the standard parametrization of the unit circle centered at the origin. Compute the following integrals:

(a) $A = \displaystyle\int_\gamma \frac{e^z}{z^2(z-2)}\,dz.$

(b) $B = \displaystyle\int_\gamma \frac{\sin(z)}{z(z-2i)}\,dz.$

(c) $C = \displaystyle\int_\gamma \frac{z^3-2z}{z(z-10)}\,dz.$

(d) $D = \int_\gamma \left(\frac{1}{z^2} + \frac{1}{z} - e^z \sin(z)\right) dz.$

5. Consider the function $f(z) = \log(1+z^2)$. What is the maximal subset of $\mathbb{C}$ on which f is holomorhic? Compute the Taylor series at $z_0 = 0$. What is it's radius of convergence?

6. In this exercise, we will prove Liouville's theorem, which states that if $f : \mathbb{C} \to \mathbb{C}$ is an entire function which is bounded (i.e. there exists some $M > 0$ such that $|f(z)| \leqslant M$ for all $z \in \mathbb{C}$), then f must be constant.

(a) Show that, for any $z_0 \in \mathbb{C}$, if $\gamma_R(t) = z_0 + Re^{it}$ for $t \in [0, 2\pi]$, then

$$|f'(z_0)| \leqslant \int_0^{2\pi} \frac{\left|f(\gamma_R(t))\right|}{R}\, dt.$$

(Hint: Use Cauchy's integral formula.)

(b) Assuming that f is bounded, show that $f'(z_0) = 0$ for all $z_0 \in \mathbb{C}$. Hence, f is constant.

7. Let $\mathcal{D} \subseteq \mathbb{C}$ be open and simply connected. Let $f : \mathcal{D} \to \mathbb{C}$ be holomorphic. For any $z_0 \in \mathcal{D}$ and any $R > 0$ such that the closed disk

$$\overline{B_R(z_0)} = \left\{ z \in \mathbb{C} : |z - z_0| \leqslant R \right\}$$

is contained inside $\mathcal{D}$, show that the following identity holds:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + Re^{it}\right) dt.$$

The above is known as the *mean value property* for holomorphic functions.
(Hint: Start from Cauchy's integral formula on a suitable curve, and calculate the integral explicitly.)

Solutions.

1. **(a)** $f(z) = e^z$ at $z_0 = 0$

The Taylor expansion of e^z at 0 is:

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

So the Taylor coefficients are $a_n = \frac{1}{n!}$.

(b) $f(z) = e^z$ at $z_0 = \pi i$

We expand around $z_0 = \pi i$ using the substitution $w = z - \pi i$:

$$f(z) = e^z = e^{\pi i} e^{z - \pi i} = -1 \cdot \sum_{n=0}^{\infty} \frac{(z - \pi i)^n}{n!}$$

So the Taylor coefficients are $a_n = \frac{(-1)}{n!}$.

(c) $f(z) = e^{z^2}$ at $z_0 = 0$

We expand e^{z^2} using the power series of the exponential:

$$f(z) = \sum_{n=0}^{\infty} \frac{(z^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{z^{2n}}{n!}$$

So:

$$a_{2n} = \frac{1}{n!}, \quad a_{2n+1} = 0$$

(d) $f(z) = \cos(z - 3)$ at $z_0 = 3$

This is just the Taylor series of cosine centered at 0, shifted:

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (z - 3)^{2n}$$

So:

$$a_{2n} = \frac{(-1)^n}{(2n)!}, \quad a_{2n+1} = 0$$

(e) $f(z) = \sin^2(z)$ at $z_0 = 0$

We use the identity:

$$\sin^2(z) = \frac{1 - \cos(2z)}{2}$$

and expand $\cos(2z)$:

$$\cos(2z) = \sum_{n=0}^{\infty} \frac{(-1)^n (2z)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n z^{2n}}{(2n)!}$$

Thus:

$$f(z) = \frac{1}{2} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n 4^n z^{2n}}{(2n)!} = \sum_{n=1}^{\infty} \left(-\frac{(-1)^n 4^n}{2(2n)!} \right) z^{2n}$$

So:

$$a_0 = 0, \quad a_{2n} = -\frac{(-1)^n 4^n}{2(2n)!} \text{ (for } n \geqslant 1), \quad a_{2n+1} = 0$$

2. **(a)** $g(z) = \frac{e^z}{(z-2)^2}$ at $z_0 = 2$

Let $w = z - 2$. Then $e^z = e^{w+2} = e^2 e^w$, so:

$$g(z) = \frac{e^z}{(z-2)^2} = \frac{e^2 e^w}{w^2} = e^2 \cdot \frac{1}{w^2} \cdot \sum_{n=0}^{\infty} \frac{w^n}{n!} = e^2 \sum_{n=0}^{\infty} \frac{w^{n-2}}{n!}$$

So the Laurent series is:

$$g(z) = e^2 \sum_{n=0}^{\infty} \frac{(z-2)^{n-2}}{n!}$$

This has a pole of order 2 at $z = 2$, so the singularity is a pole of order 2.

The radius of convergence is infinite, since there is no other singularity: e^z is entire and the only singularity comes from the denominator.

(b) $g(z) = \frac{1}{z(z+1)}$ at $z_0 = -1$

We write:

$$\frac{1}{z(z+1)} = \frac{1}{z} - \frac{1}{z+1}$$

Now expand $\frac{1}{z}$ around $z = -1$, that is, in terms of $w = z + 1$:

$$\frac{1}{z} = \frac{1}{w-1} = -\sum_{n=0}^{\infty} w^n = -\sum_{n=0}^{\infty} (z+1)^n \quad \text{for } |z+1| < 1$$

So:

$$g(z) = \frac{1}{z} - \frac{1}{z+1} = -\sum_{n=0}^{\infty} (z+1)^n - \frac{1}{z+1} = -\frac{1}{z+1} - \sum_{n=0}^{\infty} (z+1)^n$$

This is a Laurent series centered at $z = -1$, with a simple pole at $z = -1$.

The radius of convergence is 1, limited by the distance to the other singularity at $z = 0$.

(c) $g(z) = \frac{z^3-z+2}{(z-i)^2}$ at $z_0 = i$

Let $w = z - i$. The numerator is a polynomial in z, we can reexpress it as a polynomial in $z - i$ (this will automatically be a Taylor series for the numerator):

$$z^3-z+2 = (z-i+i)^3-(z-i+i)+2 = (z-i)^3+3(z-i)^2i+3(z-i)i^2+i^3-(z-i+i)+2 = (z-i)^3+3i(z-i)^2$$

We can write:

$$g(z) = \frac{f(z)}{(z-i)^2} = \frac{(z-i)^3 + 3i(z-i)^2 - 4(z-i) + (2-2i)}{(z-i)^2} = \frac{2-2i}{(z-i)^2} + \frac{-4}{z-i} + 3i + (z-i).$$

This is a Laurent expansion centered at $z = i$. The function has a pole of order 2 at $z = i$. The radius of convergence is infinite, since $g(z)$ is holomorhpic everywhere except at $z = i$.

(d) $g(z) = \frac{\cos((z-1)^2)}{(z-1)^3}$ at $z_0 = 1$

Let $w = z - 1$. Then:

$$g(z) = \frac{\cos(w^2)}{w^3}$$

Expand numerator:

$$\cos(w^2) = \sum_{n=0}^{\infty} \frac{(-1)^n w^{4n}}{(2n)!}$$

So:

$$g(z) = \sum_{n=0}^{\infty} \frac{(-1)^n w^{4n-3}}{(2n)!}$$

This is a Laurent series in powers of $w = z-1$. The lowest power is w^{-3}, so the singularity is a pole of order 3 at $z = 1$.

The radius of convergence is infinite, since $\cos(z)$ is entire so $g(z)$ has no other singularity.

3. **(a)** $f(z) = z\cos\left(\frac{1}{z}\right)$ at $z_0 = 0$

Recall the Taylor series expansion of $\cos(w)$ (centered at $w_0 = 0$):

$$\cos(w) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} w^{2n}.$$

The radius of convergence of the above series is infinite, so we can plug in any finite complex number w; we will plug in $w = \frac{1}{z}$ (recall that, for a Laurent series, we never evaluate *at* the center z_0, where the function is singular):

$$\cos\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \frac{1}{z^{2n}} = \frac{1}{z^0} - \frac{1}{2!z^2} + \frac{1}{4!z^4} - \cdots$$

Multiplying by z:

$$f(z) = z\cos\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{1-2n}$$

This gives:

$$f(z) = z - \frac{1}{2!} \cdot \frac{1}{z} + \frac{1}{4!} \cdot \frac{1}{z^3} - \cdots$$

This is a Laurent series with infinitely many negative powers of z, so the singularity at $z = 0$ is an essential singularity.

Radius of convergence: infinite, since $\cos(1/z)$ is defined and holomorphic for all $z \neq 0$.

(b) $f(z) = \frac{\sin(z)}{z-\pi}$ at $z_0 = \pi$

Let $w = z - \pi$. Then $z = \pi + w$, so:

$$f(z) = \frac{\sin(\pi + w)}{w} = \frac{-\sin(w)}{w}$$

Now expand $\sin(w)$:

$$\sin(w) = \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n+1}}{(2n+1)!} \Rightarrow \frac{\sin(w)}{w} = \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n}}{(2n+1)!}$$

Thus:

$$f(z) = -\sum_{n=0}^{\infty} \frac{(-1)^n (z-\pi)^{2n}}{(2n+1)!}$$

This is a Taylor series (no negative powers), so $z = \pi$ is a regular point (i.e., removable singularity).

Radius of convergence: infinite, since there is no other singularity ($\sin(z)$ is entire).

(c) $f(z) = \frac{z^{1/2}}{(z-1)^2}$ at $z_0 = 1$

Let $w = z - 1$, so $z = 1 + w$. Then:

$$f(z) = \frac{(1+w)^{1/2}}{w^2}$$

We expand the numerator using the binomial series:

$$(1+w)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} w^n = 1 + \frac{1}{2}w - \frac{1}{8}w^2 + \cdots$$

So:

$$f(z) = \frac{1 + \frac{1}{2}w - \frac{1}{8}w^2 + \cdots}{w^2} = \frac{1}{w^2} + \frac{1}{2w} - \frac{1}{8} + \cdots$$

This is a Laurent series with a pole of order 2 at $z = 1$.

The singularity at $z = 1$ is a pole of order 2. Radius of convergence is 1.

4. Let $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$, be the positively oriented unit circle. ***Remark.*** *This exercise is easilty solved with the residue theorem, but during the week when these exrecises were given, the residue theorem had not been taught in class yet.*

(a) $A = \int_\gamma \frac{e^z}{z^2(z-2)}\, dz$
The above integral can be written as

$$A = \int_\gamma \frac{f(z)}{z^2}\, dz, \quad f(z) = \frac{e^z}{(z-2)}.$$

Since f is holomorphic in the interior of γ, from the Cauchy integral formula we infer that

$$A = \frac{2\pi i}{1!} f'(0) = -\frac{3}{2}\pi i.$$

(b) $B = \int_\gamma \frac{\sin z}{z(z-2i)}\, dz$
We will argue similarly as before:

$$B = \int_\gamma \frac{f(z)}{z}\, dz, \quad f(z) = \frac{\sin(z)}{z-2i}.$$

Since f is holomorphic in the interior of γ, by applying Cauchy's integral formula we get

$$B = 2\pi i f(0) = 0.$$

(another way to see it: $\frac{\sin(z)}{z}$ has a regular point at $z = 0$, so the whole function iniside the integral extends holomorphically to $z = 0$; hence, by Cauchy's theorem, the integral has to be zero).

(c) $C = \int_\gamma \frac{z^3-2z}{z(z-10)}\, dz$
We begin by simplifying the integrand:

$$\frac{z^3-2z}{z(z-10)} = \frac{z(z^2-2)}{z(z-10)} = \frac{z^2-2}{z-10}, \quad \text{for } z \neq 0$$

This function $f(z) = \frac{z^2-2}{z-10}$ is holomorphic everywhere except at $z = 10$, which lies *outside* the unit circle γ. Since $f(z)$ is holomorphic on and inside the contour γ, we can apply **Cauchy's theorem**:

$$\text{If } f \text{ is holomorphic on a domain containing } \gamma, \text{ then } \int_\gamma f(z)\, dz = 0$$

Hence,

$$C = 0$$

(d) $D = \int_\gamma \left(\frac{1}{z^2} + \frac{1}{z} - e^z \sin z\right)\, dz$
We split the integral:

$$D = \int_\gamma \frac{1}{z^2}\, dz + \int_\gamma \frac{1}{z}\, dz - \int_\gamma e^z \sin z\, dz$$

- $\int_\gamma \frac{1}{z^2}\,dz = 0$ (we did this calculation in class explicitly with the parametrization; one can also verify it using Cauchy's integral formula),
- $\int_\gamma \frac{1}{z}\,dz = 2\pi i$ (same reasoning as above),
- $\int_\gamma e^z \sin z\,dz = 0$, since $e^z \sin z$ is entire (analytic everywhere), and the integral over a closed curve of a holomorphic function is zero.

Thus, we conclude:

$$D = 2\pi i.$$

5. Consider the function $f(z) = \log(1+z^2)$.

The function $f(z) = \log(1+z^2)$ is defined as a composition:

$$f(z) = \log(w), \quad \text{where } w = 1+z^2$$

The logarithm function $\log(w)$ is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$. Therefore, $f(z)$ is holomorphic wherever $1+z^2 \notin (-\infty, 0]$. If $z = x+yi$, we have $1+z^2 = 1+x^2-y^2+2xyi$. So $1+z^2 \in (-\infty, 0]$ if and only if $xy = 0$ and $1+x^2-y^2 \leqslant 0$; this can only happen when $x = 0$ and $|y| \geqslant 1$, Thus, the maximal domain on which $f(z)$ is holomorphic is:

$$\mathbb{C} \setminus \left(i(-\infty, -1] \cup i[1, \infty)\right).$$

We use the Taylor expansion of the logarithm:

$$\log(1+u) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} u^n, \quad \text{valid for } |u| < 1$$

Set $u = z^2$, so:

$$f(z) = \log(1+z^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^{2n}, \quad \text{for } |z^2| < 1 \Rightarrow |z| < 1$$

The Taylor series at $z = 0$ is therefore:

$$f(z) = z^2 - \frac{z^4}{2} + \frac{z^6}{3} - \frac{z^8}{4} + \cdots$$

The radius of convergence of the Taylor series centered at $z = 0$ is equal to the distance from 0 to the nearest singularity of the function in the complex plane. Since the domain of holomorphicity is $\mathbb{C} \setminus (i(-\infty, -1] \cup i[1, \infty))$, the radius of convergence is:

$$R = 1.$$

6. In this exercise, we will prove Liouville's theorem, which states that if $f : \mathbb{C} \to \mathbb{C}$ is an entire function which is bounded (i.e. there exists some $M > 0$ such that $|f(z)| \leqslant M$ for all $z \in \mathbb{C}$), then f must be constant.

(a) Let $z_0 \in \mathbb{C}$, and consider the circle of radius $R > 0$ centered at z_0, parametrized by:

$$\gamma_R(t) = z_0 + Re^{it}, \quad t \in [0, 2\pi]$$

By Cauchy's integral formula:

$$f'(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^2} \, dz.$$

Along the curve γ_R, we have $|z - z_0| = R$, and $\gamma_R'(t) = iRe^{it}$. Substituting in the integral (using the parametrization):

$$f'(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{R^2} \cdot iRe^{it} dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{R} e^{it} dt$$

Taking modulus and using the triangle inequality for integrals over $\mathbb{R}$ (which says that $\left| \int_a^b f(t) \, dt \right| \leqslant \int_a^b |f(t)| \, dt$):

$$|f'(z_0)| \leqslant \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f(z_0 + Re^{it})}{R} e^{it} \right| dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(z_0 + Re^{it})|}{R} dt$$

Hence,

$$\boxed{|f'(z_0)| \leqslant \int_0^{2\pi} \frac{|f(z_0 + Re^{it})|}{R} \cdot \frac{dt}{2\pi}}$$

Or simply (since $\frac{1}{2\pi} < 1$):

$$\boxed{|f'(z_0)| \leqslant \int_0^{2\pi} \frac{|f(\gamma_R(t))|}{R} \, dt}$$

(b) Now assume that f is bounded: there exists $M > 0$ such that $|f(z)| \leqslant M$ for all $z \in \mathbb{C}$. From part (a), for any $R > 0$:

$$|f'(z_0)| \leqslant \int_0^{2\pi} \frac{|f(z_0 + Re^{it})|}{R} \, dt \leqslant \int_0^{2\pi} \frac{M}{R} \, dt = \frac{2\pi M}{R}$$

Since this bound holds for all $R > 0$, we take the limit as $R \to \infty$:

$$|f'(z_0)| \leqslant \frac{2\pi M}{R} \to 0 \quad \Rightarrow \quad \boxed{f'(z_0) = 0}$$

Since $z_0 \in \mathbb{C}$ was arbitrary, this shows $f' \equiv 0$ on $\mathbb{C}$, so:

$$\boxed{f \text{ is constant on } \mathbb{C}}$$

as required.

7. Let $\mathcal{D} \subseteq \mathbb{C}$ be open and simply connected, and let $f : \mathcal{D} \to \mathbb{C}$ be holomorphic. Let $z_0 \in \mathcal{D}$, and suppose that for some $R > 0$, the closed disk

$$\overline{B_R(z_0)} = \{z \in \mathbb{C} : |z - z_0| \leqslant R\}$$

is entirely contained in $\mathcal{D}$. We want to show the identity:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{it})\, dt$$

which is called the *mean value property* for holomorphic functions.

Proof:

We apply **Cauchy's integral formula** for f on the circle of radius R centered at z_0. Let $\gamma(t) = z_0 + Re^{it}$, $t \in [0, 2\pi]$, which parametrizes the positively oriented boundary of the disk.

Cauchy's formula states:

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz$$

We now compute this integral explicitly using the parametrization $z = \gamma(t) = z_0 + Re^{it}$, so:

$$\gamma'(t) = iRe^{it}, \quad z - z_0 = Re^{it}.$$

Substituting into the integral:

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{Re^{it}} \cdot iRe^{it}\, dt$$

Now simplify:

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} f(z_0 + Re^{it}) \cdot i\, dt = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{it})\, dt$$

$$\boxed{f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{it})\, dt}$$

which is exactly the mean value property.

□